23/11/23

MATH4030 Tutorial

Announcements:

- HW5 due 27/11.

<u>Recall</u>: W is a vector field if it is a correspondence between $p \in S$ and $W_p = W(p) \in T_pS$ is smooth if we can write $W(u,v) = \alpha(u,v) X_u + \beta(u,v) X_v$ for a param. $X(u,v)$ around $p \in S$ where α, β are smooth. (Remembering that $T_pS = \text{span}\{X_u|_p, X_v|_p\}$.

We want to talk about the "derivative"/"variation" of a vector field.

But $\lim_{t \to 0} \frac{W(\alpha(t)) - W(\alpha(0))}{t}$ is not well-defined in T_pS. $\alpha(0) = p$, $\underset{t \neq 0}{\alpha(t) \neq p}$

Then $W(\alpha(t))$, $W(\alpha(0)) \in \mathbb{R}^3$

$$\underset{T_qS}{W(\alpha(t))} \neq \underset{T_pS}{W(\alpha(0))}$$

Solution is to define the covariant derivative in direction v by if α is a curve w/ $\alpha(0) = p$, $\alpha'(0) = v$ and setting

$$D_v W|_p = \left(\frac{d}{dt}\Big|_{t=0} W(\alpha(t))\right)^T \leftarrow \text{take the tangential component}$$

B/c: in general $\frac{d}{dt}\Big|_{t=0} W(\alpha(t)) = f' X_u + g' X_v + f X_u' + g X_v'$

$W = f X_u + g X_v$

$$= f' X_u + g' X_v + f(X_{uu} \cdot u' + X_{uv} \cdot v') + g(X_{vu} \cdot u' + X_{vv} \cdot v')$$

where note $X_{uu} = \Gamma^u_{uu} X_u + \Gamma^v_{uu} X_v + \mathrm{II}_{uu} N$

$X_{uv} = \dots$

$X_{vu} = \dots$

$X_{vv} = \dots$

$$= \underbrace{\sim X_u + \sim}_{\text{coefficients in terms of } \Gamma} X_v + \underbrace{\sim}_{\text{coefficients in terms of } \mathrm{II}} N.$$

a non-constant curve $\alpha : I \to S$ is a geodesic if $D_{\alpha'} \alpha' \equiv 0$.

Prop: α is a geodesic $\Rightarrow$ · α is param. by arc-length
· $k_g \equiv 0$.

ODE: α is a geodesic iff it satisfies the ODE

$$(*) \qquad u_k'' + \sum_{i,j} \Gamma_{ij}^k u_i' u_j' = 0 \quad \text{for all } k = 1, 2.$$

Geometric Characterization: a regular curve $C \subset S$ $(k \neq 0)$ is a geodesic iff its principal normal at p is parallel to the normal of S at p.

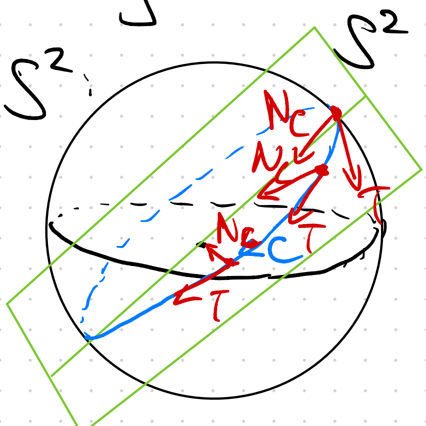
Def 8a (p. 249) of do Carmo

Q1: Part (a): Use geometric characterization to show that great circles are geodesics on S^2

(Great circles are intersections of S^2 with planes that pass through the centre)

Part (b): Use geometric characterization to find the geodesics on a cylinder (will also need to use the fact that cylinder $X(u,v) = (\cos u, \sin u, v)$ is locally isometric to uv plane)

Pf: S^2:



N_C is pointing along the line connecting the curve to the centre of S^2 and this is clearly parallel to the normal of S^2 at that point.

Uniqueness.

Cylinder:

N_C points along the line connecting the curve to the axis of the cylinder which is parallel to Normal of cylinder.

Circles, Helixes, vertical lines.

Using isometry between uv plane and cylinder given by $X(u,v) = (\cos u, \sin u, v)$, since geodesic condition is invariant w.r.t. local isometries, geodesics α on uv plane correspond to geodesics $X \circ \alpha$ on cylinder.

But geodesics on uv plane are the straight lines.

Case 1: $u(s) = s$, $v(s) = 0$ $\rightsquigarrow$ $X(s,0) = (\cos(s), \sin(s), 0)$ circle

Case 2: $u(s) = 0$, $v(s) = s$ $\rightsquigarrow$ $X(0,s) = (0, 0, s)$ vertical line

Case 3: $u(s) = as$, $v(s) = bs$, $a^2+b^2 = 1$ $\rightsquigarrow$ $X(as, bs) = (\cos as, \sin as, bs)$

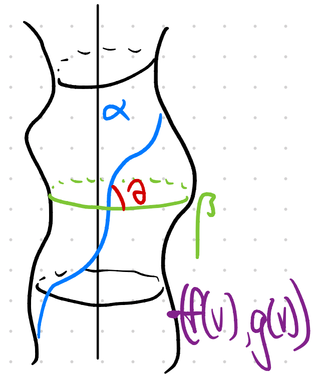
$a^2+b^2=1$ circular helix

Q2: For a surface of revolution, $X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$, $f(v)>0$

(4) becomes

$$\begin{cases} u'' + \dfrac{2ff_v}{f^2} u'v' = 0 & (1) \\[2ex] v'' - \dfrac{ff_v}{f_v^2+g_v^2}(u')^2 + \dfrac{f_vf_{vv}+g_vg_{vv}}{f_v^2+g_v^2}(v')^2 = 0 & (2). \end{cases}$$

do Carmo.



For a surface of revolution, let α be a geodesic intersecting a parallel $\beta(t) = X(t, c)$, $c=$const. at an angle θ. Let $r=$ distance to the axis of revolution. Then prove Clairaut's relation:

$$r\cos\theta = f(v(t))\cos(\theta) = \text{const.}$$

Hint: use (1).

解: $u'' + \frac{2ff_v}{f^2} u'v' = 0 \iff f^2u'' + 2ff_v u'v' = 0 \iff (f^2u')' = 0$

$\Rightarrow f^2 u' = \text{const.}$

$\beta(t) = X(t,c)$, $\beta'(t) = X_u$, $\alpha(t) = X(u(t), v(t))$

$\alpha'(t) = X_u \cdot u'(t) + X_v \cdot v'(t)$.

$$\cos\theta = \frac{\langle \alpha'(t), \beta'(t)\rangle}{\cancel{|\alpha'(t)|}\,|\beta'(t)|} = \frac{\langle X_u u' + X_v v', X_u\rangle}{|X_u|}$$

$$= \frac{u'\underbrace{|X_u|^2}_{=f^2} + v'\overbrace{\langle \cancel{X_u}, X_v\rangle}^{0}}{\underbrace{|X_u|}_{=f}} = f u'$$

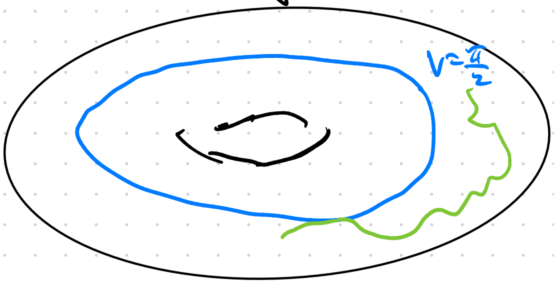
so then since f gives distance to axis, we have $r\cos\theta = f^2 u' = \text{const.}$

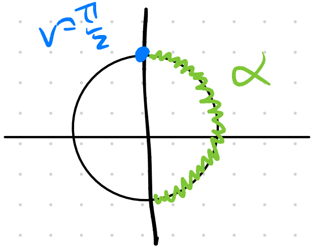
Converse: if $r\cos\theta = \text{const}$, then either α is a parallel or is a geodesic.
Pressley (Elementary DG p. 183-185).

Application of Clairaut's Relation: For the torus T param. by

$$X(u,v) = ((r\cos v + a)\cos u, (r\cos v + a)\sin u, r\sin v), \quad 0 < r < a$$

If a geodesic is tangent to the parallel $v = \frac{\pi}{2}$ at some point t_0, then it is entirely contained in the region of T given by $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.





Can also classify and analyze behavior of geodesics on the torus T.